On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces (Cohomology of Finite Groups and Related Topics)

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On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces

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Abstract

1 notations

Let p be a prime and $\mathbb{F}_q$ be the finite field with q elements. Let $G_{\mathbb{Z}}$ be a Chevalley group scheme such that $\mathbb{C}$-rational points $G_{\mathbb{Z}}(\mathbb{C})$ is a simply connected complex Lie group when we change its topology. Hereafter we denote $G_{\mathbb{Z}}(K)$ (resp. $G_{\mathbb{Z}}(\mathbb{C})$) by $G(K)$ (resp. G) for a field K. We also denote its classifying space by $\mathrm{B}G$ and define the free loop space $\mathcal{L}\mathrm{B}G$ of $\mathrm{B}G$ and the loop space $\Omega\mathrm{B}G$ of $\mathrm{B}G$ by

$$\mathcal{L}\mathrm{B}G = \{l \mid l\colon S^1 \to \mathrm{B}G\} \quad \text{and} \quad \Omega\mathrm{B}G = \{l \mid l(1) = *, l \in \mathcal{L}\mathrm{B}G\},$$

where S^1 is the unit circle on the complex number $\mathbb{C}$ and $*$ is a base point of $\mathrm{B}G$. It is well known that $\Omega\mathrm{B}G$ is weakly homotopy equivalent to G.

2 results and comments

Theorem . *Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements and l be a prime number that divides $q-1$ but does not divide the order of the Weyl group of G. Then we have an ring isomorphism*

$$H^*(\mathcal{L}\mathrm{B}G, \mathbb{Z}/l) \cong H^*(G(\mathbb{F}_q), \mathbb{Z}/l) \cong H^*(\mathrm{B}G, \mathbb{Z}/l) \otimes H^*(G, \mathbb{Z}/l).$$

We can prove the theorem immediately from Kleinerman [3] and Kono-Kozima [4].

Remark . Our theorem is partial. Here we indicate an example.

Theorem (Fong-Milgram [1],Kono-Kozima [4]). *Let G_2 be an exceptional Lie type G_2. Then we have a ring isomorphism*

$$H^*(\mathcal{L}BG_2, \mathbb{Z}/2) \cong H^*(G_2(\mathbb{F}_q), \mathbb{Z}/2)$$

for $4|q-1$.

We propose a question : Let l be a prime number such that l (resp. 4) divides $q-1$ if l is odd (resp. even). Then we have a ring isomorphism

$$H^*(\mathcal{L}BG, \mathbb{Z}/l) \cong H^*(G(\mathbb{F}_q), \mathbb{Z}/l)$$

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